

p -Nuclearity in a New Perspective

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Abstract

In this paper we try to settle some confused points concerning the use of the notion of p -nuclearity in the mathematical and physical literature, pointing out that the nuclearity index in the physicists' sense vanishes for any $p > 1$. Our discussion of these issues suggests a new perspective, in terms of ε -entropy and operator spaces, which might permit connections to be drawn between phase space criteria and quantum energy inequalities.

1 Introduction

The Araki–Haag–Kastler programme of algebraic quantum field theory [16] seeks to describe theories in terms of algebras $\mathfrak{A}(\mathcal{O})$ of observables associated with open bounded regions \mathcal{O} in spacetime, with particular regard to their net structure encoded by the map $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$. The structural analysis of general quantum field theories within this framework proceeds from a small number of axioms relating to Poincaré covariance and causality. However, these axioms by themselves do not guarantee reasonable physical behaviour, such as the existence of thermodynamical equilibrium states or particle-like excitations. For these properties, it turns out that one must impose additional restrictions on the phase space volume available to the theory, according to some suitable notion of size. Criteria of this type were first

introduced by Haag and Swieca [17] in terms of a compactness condition, and have since proved their worth in various contexts. In particular, the nuclearity criterion of Buchholz and Wichmann [10] was applied to the analysis of thermodynamic properties, *e. g.* in [8], and to modular theory [6].

A typical (but by no means the only) setting for these criteria is the following. Two Banach spaces \mathfrak{E} and \mathfrak{F} are identified, along with a class of continuous maps $\Theta_{\beta, \mathcal{O}} : \mathfrak{E} \rightarrow \mathfrak{F}$ which are associated with energetically damped local excitations of the vacuum, localised in \mathcal{O} and with the damping parametrised by the inverse temperature β . The phase space requirement is then encoded by demanding that these maps be ‘approximately finite rank’: more precisely, that they belong to some class of operators containing the finite rank maps, and contained within the class of compact maps from \mathfrak{E} to \mathfrak{F} . A numerical index, ν , is defined on this class of operators, and the asymptotic behaviour of $\nu(\Theta_{\beta, \mathcal{O}})$ as $\beta \rightarrow 0^+$ may also be constrained as part of the nuclearity criterion. The main purpose of this paper is to draw attention to a serious shortcoming of one such index, namely the so-called p -nuclearity index. Indeed, we will show that this index vanishes identically for $p > 1$!

The starting point for our discussion is the investigation of [6], in which mappings of type l^p are considered¹ and, using the result [20, 8.4.2 Proposition], seen to be nuclear for $0 < p \leq 1$. In fact, there exists a decomposition of a mapping $\Theta : \mathfrak{E} \rightarrow \mathfrak{F}$ of type l^p (\mathfrak{E} and \mathfrak{F} normed vector spaces) in terms of sequences of vectors $\{\varphi_k : k \in \mathbb{N}\} \subset \mathfrak{F}$ and of continuous linear functionals $\{\ell_k : k \in \mathbb{N}\} \subset \mathfrak{E}^*$ such that

$$\Theta(x) = \sum_{k=1}^{\infty} \ell_k(x) \varphi_k, \quad x \in \mathfrak{E}, \quad (1.1a)$$

$$\text{and} \quad \sum_{k=1}^{\infty} \|\ell_k\|^p \|\varphi_k\|^p < \infty. \quad (1.1b)$$

The corresponding number

$$\|\Theta\|_p \doteq \inf \left(\sum_{k=1}^{\infty} \|\ell_k\|^p \|\varphi_k\|^p \right)^{\frac{1}{p}}, \quad (1.1c)$$

defines a quasi-norm on the set of these mappings which is called the p -norm, where the infimum extends over all possible decompositions of Θ . Now the map Θ is said to be nuclear if $\|\Theta\|_1$ is finite, and, since $\|\Theta\|_p \geq \|\Theta\|_1$ for $0 < p \leq 1$, the l^p maps are indeed nuclear for p in this interval.

¹See Sec. 2 for the definition of this class

For $p \leq 1$ the above definition is sound and indeed has an intuitive interpretation in algebraic QFT. Moreover, its use has led to deep insights as already mentioned. However, later authors have often adopted the above notions without any restriction on p , under the label of *p-nuclearity*. The first occurrence may be in [9, Section 2], further examples of this usage may be found in [5, 4, 19, 11]. But it ought to be emphasised that the physically relevant results published in these papers depend on the case $p \leq 1$ and therefore remain valid, despite the problems with $p > 1$ we will point out. The explicit definition of this version, following Buchholz, D’Antoni, and Longo in [6] (who, however, did *not* use the term *p-nuclearity* in this context) can be formulated as follows.

Definition 1.1. Let \mathfrak{E} and \mathfrak{F} be normed vector spaces. An operator $\Theta : \mathfrak{E} \rightarrow \mathfrak{F}$ is called ***p-nuclear***, $p > 0$, if there exist sequences of vectors $\{\varphi_k : k \in \mathbb{N}\} \subset \mathfrak{F}$ and of continuous linear functionals $\{\ell_k : k \in \mathbb{N}\} \subset \mathfrak{E}^*$ such that

$$\Theta(x) = \sum_{k=1}^{\infty} \ell_k(x) \varphi_k, \quad x \in \mathfrak{E}, \quad (1.2a)$$

$$\text{and} \quad \sum_{k=1}^{\infty} \|\ell_k\|^p \|\varphi_k\|^p < \infty. \quad (1.2b)$$

For later reference we call a combination of functionals and vectors satisfying the relations (1.2a) and (1.2b) a ***p-nuclear decomposition of Θ*** . The *p-nuclearity index* of the operator Θ is defined by

$$\|\Theta\|_p \doteq \inf_{\substack{p\text{-nuclear} \\ \text{decompositions}}} \left(\sum_{k=1}^{\infty} \|\ell_k\|^p \|\varphi_k\|^p \right)^{\frac{1}{p}}. \quad (1.2c)$$

Now it is true that mappings of type l^p are *p-nuclear* for arbitrary $p > 0$, as we will show in the next section by reworking Pietsch’s argument. But two important caveats should be borne in mind. First, there is a notion of *p-nuclearity* for $p \geq 1$ in the mathematical literature which differs from that given above (to which we shall refer as the physicists’ definition) except in the case $p = 1$. Second, the *p-nuclearity index* of (1.2c) can easily be seen to vanish for any *p-nuclear* operator if $p > 1$. Furthermore, if \mathfrak{E} is a Banach space with a Schauder basis then every bounded operator from \mathfrak{E} to \mathfrak{F} is *p-nuclear* for all $p > 1$ (with necessarily vanishing *p-nuclearity index*). To the best of our knowledge this has not been pointed out before.

The object of this letter is to clarify the above issues and also to indicate a possible remedy for the problem just mentioned, along with alternative directions

for research. Our investigation was inspired by the wish to use phase space criteria like nuclearity to establish *quantum energy inequalities* (QEIs). These are state-independent lower bounds on weighted averages of the stress-energy tensor, which have been established for various free field theories and two-dimensional conformal field theory (see [14] and references therein). Now QEIs are a manifestation of the uncertainty principle, and therefore intimately related to phase space properties of the theory (see [13] for quantum mechanical examples of this connection). It is therefore natural to enquire whether there is a more formal connection between QEIs and nuclearity criteria. Some progress on this question has already been made, in the context of generalised free fields with discrete mass spectrum, and will be reported in full elsewhere. It turns out that the existence of QEIs with reasonable scaling behaviour are equivalent to growth conditions on the mass spectrum which are sufficient for nuclearity to hold with the correct asymptotic behaviour of the nuclearity index. In order to establish full equivalence between QEIs and nuclearity, it is necessary to obtain lower bounds on the nuclearity index. [Upper bounds are of course provided by any decomposition entering the definition of the p -norm in (1.2c).] As a first step in this direction we indicate an *exact* expression for the 2-nuclearity index of a 2-nuclear operator acting between Hilbert spaces, using a modified notion of 2-nuclearity. Another possible route from phase space criteria to quantum inequalities could be the use of the notion of ε -entropy (ε -content) in the context of operator spaces. Not only upper but also lower bounds on the ε -entropy can be defined in the limit of small ε [1]. It is hoped to return to these issues elsewhere.

2 Decompositions of Mappings of Type l^p for Arbitrary p and Related Problems

We begin with the formal definition of mappings of type l^p .

Definition 2.1 (Pietsch [20, 8.1.1]). Let \mathfrak{E} and \mathfrak{F} be normed vector spaces. For an arbitrary continuous operator $\Theta : \mathfrak{E} \rightarrow \mathfrak{F}$ we define the *k 'th approximation number* $\alpha_k(\Theta)$, $k \in \mathbb{N}_0$, through

$$\alpha_k(\Theta) \doteq \inf \{ \|\Theta - \Theta_k\| : \Theta_k \text{ an operator of at most finite rank } k \}. \quad (2.1)$$

These approximation numbers can now be used to define certain subspaces $l^p(\mathfrak{E}, \mathfrak{F})$ of continuous operators.

Definition 2.2 (Pietsch [20, 8.2.1]). A continuous operator $\Theta : \mathfrak{E} \rightarrow \mathfrak{F}$ is said to be a *mapping of type l^p* (called *p -approximable operator* in [18, 19.8]), $0 < p < \infty$,

if summation of the p 'th power of all approximation numbers yields a finite result:

$$\sum_{k=0}^{\infty} \alpha_k(\Theta)^p < \infty. \quad (2.2)$$

Furthermore one defines the real number

$$\rho_p(\Theta) \doteq \left(\sum_{k=0}^{\infty} \alpha_k(\Theta)^p \right)^{\frac{1}{p}}. \quad (2.3)$$

We now generalise the result of Pietsch [20, 8.4.2 Proposition] mentioned above to all $p > 0$, re-writing his proof in a notation more familiar to physicists.

Proposition 2.3. *For all $p > 0$ each mapping $\Theta \in l^p(\mathfrak{E}, \mathfrak{F})$ can be represented as*

$$\Theta(x) = \sum_{k=1}^{\infty} \lambda_k \ell_k(x) \phi_k, \quad x \in \mathfrak{E}, \quad (2.4)$$

with normalised sequences of vectors $\{\phi_k : k \in \mathbb{N}\} \subset \mathfrak{F}$ and of continuous linear functionals $\{\ell_k : k \in \mathbb{N}\} \subset \mathfrak{E}^*$ as well as of numbers $0 < \lambda_k \leq \|\Theta\|$ such that

$$\left(\sum_{k=0}^{\infty} \lambda_k^p \right)^{\frac{1}{p}} \leq 2^{2+\frac{3}{p}} \rho_p(\Theta). \quad (2.5)$$

Proof. Consider approximations Θ_n of rank $2^n - 2$ which satisfy $\|\Theta - \Theta_n\| \leq 2\alpha_{2^n-2}(\Theta)$. (Note that $\Theta_1 = 0$.) Then $\Psi_n \doteq \Theta_{n+1} - \Theta_n$ is an operator of rank $\leq 2^{n+2}$ with

$$\|\Psi_n\| \leq \|\Theta - \Theta_n\| - \|\Theta - \Theta_{n+1}\| \leq 4\alpha_{2^n-2}(\Theta),$$

since the sequence of approximation numbers is monotone decreasing, so

$$\text{rank}(\Psi_n) \|\Psi_n\|^p \leq 2^{n+2} 4^p \alpha_{2^n-2}(\Theta)^p = 2^{2p+n+2} \alpha_{2^n-2}(\Theta)^p.$$

Now the monotone decrease of the approximation numbers permits us to use Cauchy's condensation trick to write

$$\sum_{n=1}^{\infty} 2^{n-1} \alpha_{2^n-2}(\Theta)^p \leq \sum_{n=1}^{\infty} \sum_{r=2^{n-1}-1}^{2^n-2} \alpha_r(\Theta)^p = \sum_{r=0}^{\infty} \alpha_r(\Theta)^p = \rho_p(\Theta)^p.$$

Hence

$$\sum_{n=1}^{\infty} \text{rank}(\Psi_n) \|\Psi_n\|^p \leq 2^{2p+3} \sum_{n=1}^{\infty} 2^{n-1} \alpha_{2^n-2}(\Theta)^p \leq 2^{2p+3} \rho_p(\Theta)^p.$$

According to [20, 8.4.1, Lemma 2] Ψ_n as an operator of finite rank can be written as

$$\Psi_n(x) = \sum_{i=1}^{\text{rank}(\Psi_n)} \lambda_i^{(n)} \ell_i^{(n)}(x) \varphi_i^{(n)}, \quad x \in \mathfrak{E},$$

with $0 < \lambda_i^{(n)} \leq \|\Psi_n\|$ and normalised functionals $\ell_i^{(n)}$ and vectors $\varphi_i^{(n)}$ in \mathfrak{E}^* and \mathfrak{F} , respectively. Moreover,

$$\sum_{n=1}^{\infty} \sum_{i=1}^{\text{rank}(\Psi_n)} \left(\lambda_i^{(n)} \right)^p \leq \sum_{n=1}^{\infty} \text{rank}(\Psi_n) \|\Psi_n\|^p \leq 2^{2p+3} \rho_p(\Theta)^p. \quad (2.6)$$

By definition,

$$\Theta(x) = \lim_{m \rightarrow \infty} \Theta_m(x) = \lim_{m \rightarrow \infty} \sum_{n=1}^m \Psi_n(x) = \sum_{n=1}^{\infty} \sum_{i=1}^{\text{rank}(\Psi_n)} \lambda_i^{(n)} \ell_i^{(n)}(x) \varphi_i^{(n)}, \quad x \in \mathfrak{E}, \quad (2.7)$$

which in connection with (2.6) establishes the Proposition, without any restriction on p . \square

Since we are free to choose functionals and vectors that are not normalised in the representation (2.4) of the operator Θ , absorbing the coefficients λ_k into one or both of them, the above result shows that in this case the product of the norms raised to the p 'th power is summable with the bound given in (2.5). Thus we have the following Corollary.

Corollary 2.4. *Every operator $\Theta \in l^p(\mathfrak{E}, \mathfrak{F})$ is p -nuclear in the sense of Definition 1.1 for arbitrary $p > 0$.*

Despite this close relationship with mappings of type l^p , the notion of p -nuclearity as defined in Def. 1.1 is problematic in two ways. First of all it is in conflict with the mathematicians' notion that is defined for $1 \leq p \leq \infty$. For completeness we present the formal definition following Jarchow's book [18] and adopt a notation hopefully better accessible for mathematical physicists.

Definition 2.5. Let \mathfrak{E} and \mathfrak{F} be normed vector spaces. An operator $\Theta : \mathfrak{E} \rightarrow \mathfrak{F}$ is called **p -nuclear**, $1 \leq p \leq \infty$, if there exist sequences of vectors $\{\varphi_k : k \in \mathbb{N}\} \subset \mathfrak{F}$ and of continuous linear functionals $\{\ell_k : k \in \mathbb{N}\} \subset \mathfrak{E}^*$ such that

$$\Theta(x) = \sum_{k=1}^{\infty} \ell_k(x) \varphi_k, \quad x \in \mathfrak{E}, \quad (2.8)$$

and such that the sequences comply with the following additional assumptions. Again we refer to such a decomposition as a ***p-nuclear decomposition of Θ*** in each case.

(a) For $1 < p < \infty$ there hold

$$\sum_{k=1}^{\infty} \|\ell_k\|^p < \infty \quad \text{and} \quad (2.9a)$$

$$\sum_{k=1}^{\infty} |\langle \alpha | \varphi_k \rangle|^{p^*} < \infty, \quad \alpha \in \mathfrak{F}^*. \quad (2.9b)$$

Here p^* is the conjugate number to p , i. e., $p^* \doteq \frac{p}{p-1}$ so that $p^{-1} + p^{*-1} = 1$. The p -nuclearity index $v_p(\Theta)$ is then defined by

$$v_p(\Theta) \doteq \inf_{\substack{p\text{-nuclear} \\ \text{decompositions}}} \left(\left(\sum_{k=1}^{\infty} \|\ell_k\|^p \right)^{\frac{1}{p}} \cdot \sup_{\substack{\alpha \in \mathfrak{F}^* \\ \|\alpha\| \leq 1}} \left(\sum_{k=1}^{\infty} |\langle \alpha | \varphi_k \rangle|^{p^*} \right)^{\frac{1}{p^*}} \right). \quad (2.9c)$$

(b) For $p = 1$ ($1^* = \infty$) the additional conditions on the sequences are

$$\sum_{k=1}^{\infty} \|\ell_k\| < \infty \quad \text{and} \quad (2.10a)$$

$$\sup_{k \in \mathbb{N}} \|\varphi_k\| < \infty, \quad (2.10b)$$

with the 1-nuclearity index $v_1(\Theta)$ defined by

$$v_1(\Theta) \doteq \inf_{\substack{1\text{-nuclear} \\ \text{decompositions}}} \left(\left(\sum_{k=1}^{\infty} \|\ell_k\| \right) \cdot \sup_{k \in \mathbb{N}} \|\varphi_k\| \right). \quad (2.10c)$$

(c) For $p = \infty$ ($\infty^* = 1$) the conditions on the sequences are

$$\lim_{k \rightarrow \infty} \|\ell_k\| = 0 \quad \text{and} \quad (2.11a)$$

$$\sum_{k=1}^{\infty} |\langle \alpha | \varphi_k \rangle| < \infty, \quad \alpha \in \mathfrak{F}^*, \quad (2.11b)$$

with the ∞ -nuclearity index $v_{\infty}(\Theta)$ defined as

$$v_{\infty}(\Theta) \doteq \inf_{\substack{\infty\text{-nuclear} \\ \text{decompositions}}} \left(\left(\sup_{k \in \mathbb{N}} \|\ell_k\| \right) \cdot \sup_{\substack{\alpha \in \mathfrak{F}^* \\ \|\alpha\| \leq 1}} \left(\sum_{k=1}^{\infty} |\langle \alpha | \varphi_k \rangle| \right) \right). \quad (2.11c)$$

Note that the definitions (2.9c) and (2.11c) indeed yield finite values for the nuclearity index, since

$$\sup_{\substack{\alpha \in \mathfrak{F}^* \\ \|\alpha\| \leq 1}} \left(\sum_{k=1}^{\infty} |\langle \alpha | \varphi_k \rangle|^p \right) < \infty$$

for any $p > 0$ by a uniform boundedness argument (cf. [18, Section 16.5] for the case $p = 1$).

In the case $p = 1$ both notions of nuclearity coincide.

Proposition 2.6. *The nuclearity index for $p = 1$ calculated according to (1.2c) and (2.10c) yields the same result.*

Proof. (i) Let $\{\varphi_k : k \in \mathbb{N}\} \subset \mathfrak{F}$ and $\{\ell_k : k \in \mathbb{N}\} \subset \mathfrak{E}^*$ be sequences of vectors and of continuous linear functionals, respectively, complying with relations (1.2a) and (1.2b) for $p = 1$. A simple redefinition by $\ell'_k \doteq \|\varphi_k\| \ell_k$ and $\varphi'_k \doteq \|\varphi_k\|^{-1} \varphi_k$ yields another 1-nuclear decomposition of Θ in terms of unit vectors satisfying

$$\sum_{k=1}^{\infty} \|\ell_k\| \|\varphi_k\| = \sum_{k=1}^{\infty} \|\ell'_k\| \|\varphi'_k\| = \sum_{k=1}^{\infty} \|\ell'_k\|.$$

From this relation we not only infer that the nuclearity index $\|\Theta\|_1$ can be calculated by considering only such 1-nuclear decompositions in terms of sequences of unit vectors. Furthermore, these special decompositions comply with the requirements stated in Definition 2.5(b) with $\sup_{k \in \mathbb{N}} \|\varphi'_k\| = 1$ so that, according to (2.10c),

$$v_1(\Theta) \leq \sum_{k=1}^{\infty} \|\ell'_k\|.$$

Thus we conclude that

$$v_1(\Theta) \leq \|\Theta\|_1.$$

(ii) Now, let $\{\varphi_k : k \in \mathbb{N}\} \subset \mathfrak{F}$ and $\{\ell_k : k \in \mathbb{N}\} \subset \mathfrak{E}^*$ be sequences of vectors and of continuous linear functionals, respectively, complying with equation (2.8) and the requirements (2.10a) and (2.10b). Then,

$$\sum_{k=1}^{\infty} \|\ell_k\| \|\varphi_k\| \leq \left(\sum_{k=1}^{\infty} \|\ell_k\| \right) \cdot \sup_{k \in \mathbb{N}} \|\varphi_k\| < \infty,$$

so that these sequences conform to (1.2b) in Definition 1.1 for $p = 1$. Moreover, we conclude from this relation in connection with (1.2c) and (2.10c) that

$$\|\Theta\|_1 \leq v_1(\Theta).$$

Combining both results we arrive at the desired statement

$$\|\Theta\|_1 = v_1(\Theta)$$

which is valid for any operator Θ complying with either Definition 1.1 for $p = 1$ or with Definition 2.5(b). \square

But the conflict with the mathematicians' concept is not the only problem connected with Definition 1.1. In fact as it stands the corresponding p -nuclearity index is identically zero for $p > 1$, due to the following reasoning. Since no further restriction is imposed on the sequence of vectors $\{\varphi_k : k \in \mathbb{N}\} \subset \mathfrak{F}$ appearing in (1.2a), we are free to replace every term in this sum by m equal terms consisting in the product of $\ell_k(x)$ with $m^{-1}\varphi_k$. The result is another p -nuclear decomposition for Θ . But every term $\|\ell_k\|^p \|\varphi_k\|^p$ in (1.2b) is now replaced by m identical terms $\|\ell_k\|^p \|m^{-1}\varphi_k\|^p = m^{-p} \|\ell_k\|^p \|\varphi_k\|^p$. In this way, the sum in (1.2b) is multiplied by the factor $m \cdot m^{-p} = m^{-(p-1)}$ yielding another upper bound for the p -nuclearity index defined in (1.2c) which is the original one times $m^{-\frac{p-1}{p}}$. Since we are free to choose an arbitrary large natural number m , we thus get arbitrarily low upper bounds for the p -nuclearity index defined according to (1.2c) in the parameter range $1 < p < \infty$. In fact, $\|\Theta\|_p = 0$, which gives no insight into the geometrical structure of $\Theta(\mathfrak{E}_1)$ as was originally hoped. For the range $0 < p \leq 1$ no such problem with the p -nuclearity index $\|\cdot\|_p$ arises; and on the bounds of the index $v_p(\cdot)$ according to (2.9c) in Definition 2.5(a) the artificial splitting of individual terms in a given p -nuclear decomposition has no effect at all.

The above procedure can be used to produce an even more striking result when the pre-image \mathfrak{E} of the operator Θ is an infinite-dimensional Banach space with Schauder basis (in contradistinction to a Hamel basis), *i. e.*, when every element of \mathfrak{E} has a decomposition $x = \sum_{k=1}^{\infty} \alpha_k x_k$ in terms of a fixed sequence $\{x_k\}_{k \in \mathbb{N}} \subseteq \mathfrak{E}$ with a *unique* set of coefficients $\{\alpha_k\}_{k \in \mathbb{N}}$ [22].

Proposition 2.7. *Every bounded operator Θ mapping the Banach space \mathfrak{E} with Schauder basis $\{x_k\}_{k \in \mathbb{N}}$ into the normed space \mathfrak{F} is p -nuclear for every $p > 1$ in the sense of Definition 1.1.*

Proof. Let $\{\varphi_k\}_{k \in \mathbb{N}}$ denote the sequence of images of elements of the Schauder basis in \mathfrak{E} under Θ : $\varphi_k \doteq \Theta(x_k)$, $k \in \mathbb{N}$. Let furthermore $\{\ell_k\}_{k \in \mathbb{N}} \subseteq \mathfrak{E}^*$ denote the sequence of associated (continuous) coefficient functionals defined via $\ell_k(x) \doteq \alpha_k$, $k \in \mathbb{N}$, $x \in \mathfrak{E}$, where $x = \sum_{k=1}^{\infty} \alpha_k x_k$ [24, Definition 3.1 and Theorem 3.1]. By linearity and continuity of Θ , we then get the following decomposition of $\Theta(x)$:

$$\Theta(x) = \sum_{k=1}^{\infty} \ell_k(x) \Theta(x_k) = \sum_{k=1}^{\infty} \ell_k(x) \varphi_k, \quad x \in \mathfrak{E}. \quad (2.12)$$

While being a decomposition of the desired kind (1.2a), the validity of relation (1.2b) is not guaranteed. But this can be achieved by use of the same sort of dilution argument that was already applied in the paragraph preceding Proposition 2.7. Since p is supposed to be greater than 1 we can exhibit for every $k \in \mathbb{N}$ a natural number m_k such that

$$m_k^{1-p} \|\ell_k\|^p \|\phi_k\|^p < \frac{1}{k^2}. \quad (2.13)$$

Replacing the k 'th term in (2.12) by m_k identical copies $m_k^{-1} \ell_k(x) \phi_k$ we get another decomposition of $\Theta(x)$ compliant with (1.2a); but now in addition (1.2b) is satisfied since, according to (2.13),

$$\sum_{k=1}^{\infty} m_k \|m_k^{-1} \ell_k\|^p \|\phi_k\|^p = \sum_{k=1}^{\infty} m_k^{1-p} \|\ell_k\|^p \|\phi_k\|^p < \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} < \infty. \quad (2.14)$$

Thus, Θ turns out to be p -nuclear for every $p > 1$. \square

Again we perceive that the notion of p -nuclearity as formulated in Definition 1.1 loses its discriminatory power for $p > 1$.

A natural first reaction to these problems is to insist that the sequences of vectors occurring in p -nuclear decompositions should be linearly independent. However, this does not provide a remedy, as we now show. Suppose that the span of the vectors ϕ_j appearing in the nuclear decomposition of Θ has infinite codimension in \mathfrak{F} and choose countably many sequences of countably many vectors $\xi_{r,s}$, $r, s \in \mathbb{N}$, which are linearly independent of each other and the ϕ_j . The aim is to show that we can modify the decomposition of Θ by using only the $\xi_{1,s}$'s, so that the upper bound on $\|\Theta\|_p$ is reduced by a factor strictly less than one (independent of the ξ 's). By repeating this, using the $\xi_{2,s}$'s *etc.*, the upper bound becomes arbitrarily close to zero. To do this, choose α so that $\frac{1}{2} < \alpha < 2^{-\frac{1}{p}}$ which is possible for $p > 1$. We may assume without loss of generality that the $\xi_{1,s}$'s have been normalised so that

$$\Phi_{j,\pm} \doteq \frac{1}{2} \phi_j \pm \xi_{1,j}$$

have norms $\|\Phi_{j,\pm}\| \leq \alpha \|\phi_j\|$. Now replace the j 'th term in the decomposition of $\Theta(x)$ by the two terms $\ell_j(x) \Phi_{j,+} + \ell_j(x) \Phi_{j,-}$. This yields a new decomposition of Θ with linearly independent vectors and the upper bound is now multiplied by the factor $2^{\frac{1}{p}} \alpha < 1$. Continuing this procedure, the bound on the p -nuclearity index can be made arbitrarily small. Thus in particular all finite rank operators would have vanishing p -nuclearity index for $p > 1$. This argument would not be possible, of course, if we also insisted that the vectors in p -nuclear decompositions should

belong to the range of the operator concerned, and it is possible that this might yield a viable notion of p -nuclearity.

On the other hand, the actual use that is made of the notion of nuclearity in the literature, cf. e. g. [10, 7, 9, 11], hints at a slightly different solution of the problems just indicated when Hilbert spaces are considered. The calculation of upper bounds on the nuclearity index is always performed by falling back on an orthonormal basis. So a further possible attempt to overcome the difficulties for $p > 1$ is to allow only nuclear decompositions in terms of an orthonormal basis. In this case the 2-nuclearity index of an operator $\Theta : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, $\mathcal{H}_1, \mathcal{H}_2$ Hilbert spaces, is just the trace of the operator $\Theta\Theta^*$.

Proposition 2.8. *Let \mathcal{H}_1 and \mathcal{H}_2 be two Hilbert spaces and let $\Theta : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a continuous operator. Then Θ is a 2-nuclear operator in the sense of Definition 1.1 with only decompositions in terms of orthonormal bases allowed if and only if $\Theta\Theta^* : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ belongs to the trace-class. The corresponding nuclearity index is given by*

$$\|\Theta\|_2^2 = \text{Tr}(\Theta\Theta^*). \quad (2.15)$$

Proof. According to the definition there is an orthonormal basis $\{\Phi_k\}_{k \in \mathbb{N}}$ such that

$$\begin{aligned} \Theta(x) &= \sum_{k=1}^{\infty} \langle \Phi_k | \Theta(x) \rangle \Phi_k = \sum_{k=1}^{\infty} \langle \Theta^* \Phi_k | x \rangle \Phi_k, \quad x \in \mathcal{H}_1, \\ \text{and} \quad \sum_{k=1}^{\infty} \|\Theta^* \Phi_k\|^2 &< \infty. \end{aligned}$$

But the expression on the left-hand side of the last inequality is just the trace of the operator $\Theta\Theta^*$, independent of the chosen orthonormal basis. On the other hand any operator Θ with $\Theta\Theta^*$ lying in the trace-class allows for a 2-nuclear decomposition in the sense of this Proposition. The equation (2.15) is then an immediate consequence. \square

The above discussion urges us to look for a more refined notion of nuclearity to be applied to questions of physical interest. One possible direction will be indicated in the next section, inspired by our need for lower bounds on the nuclearity index in order to establish quantum energy inequalities. It is worth noting that Schumann [23] found applications of ($p > 1$)-nuclearity (according to the mathematicians' definition) to the physical question of statistical independence in quantum field theory, and it would be of interest to reproduce these results with the modified concept of nuclearity we envisage. However, one should be aware that even in [23]

the notions of p -nuclearity and p -approximability are not always carefully distinguished. We also note that Theorem 2.7 of [23] is trivial for $p > 2$, as the quantity estimated can be made arbitrarily small by a variant of the dilution argument given above.

3 ε -Entropy and Operator-Partition of Unity

To proceed further in the direction towards establishing a close relationship between the nuclearity condition and quantum energy inequalities, one requires the following:

- i) a satisfactory understanding at deeper levels of the natural reasons for the necessity to choose linearly independent vectors in \mathfrak{E} and to pick up vectors belonging to the image set in \mathfrak{F} of the nuclear map $\Theta : \mathfrak{E} \rightarrow \mathfrak{F}$, and,
- ii) sufficient control over lower bounds on an appropriate nuclearity index, as well as the upper bounds typically discussed in the literature.

For this purpose, it appears promising to introduce such viewpoints as a) the notion of ε -entropy and b) the operator partition of unity in the general context of *rigged modules* over (possibly non-selfadjoint) operator algebras formulated in the theory of operator spaces.

a) According to [1], the ε -entropy of a compact positive operator $K : \mathcal{H} \rightarrow \mathcal{H}$ on a Hilbert space \mathcal{H} can be formulated as follows: because of the compactness, the image in \mathcal{H} of the unit ball \mathcal{H}_1 under K can be covered by a finite ε -covering: $K\mathcal{H}_1 \subseteq \bigcup_{i=1}^N B(x_i; \varepsilon)$, whose minimum cardinality is denoted by $N(K, \varepsilon)$. Then, the ε -entropy $S(K, \varepsilon)$ of the compact positive operator K is defined by

$$S(K, \varepsilon) \doteq \log_2(N(K, \varepsilon)). \quad (3.1)$$

The upper and lower growth orders, $D(K)$ and $d(K)$, are defined, respectively, by

$$D(K) \doteq \overline{\lim}_{\varepsilon \searrow 0} \frac{\log S(K, \varepsilon)}{\log(1/\varepsilon)}; \quad d(K) \doteq \lim_{\varepsilon \searrow 0} \frac{\log S(K, \varepsilon)}{\log(1/\varepsilon)}, \quad (3.2)$$

which means that $S(K, \varepsilon)$ is asymptotically bounded from above and below, respectively, by $e^{D(K) \log(1/\varepsilon)}$ and $e^{d(K) \log(1/\varepsilon)}$:

$$e^{d(K) \log(1/\varepsilon)} \underset{\varepsilon \searrow 0}{\lesssim} S(K, \varepsilon) \underset{\varepsilon \searrow 0}{\lesssim} e^{D(K) \log(1/\varepsilon)}; \quad (3.3)$$

$$d(K) \log(1/\varepsilon) \underset{\varepsilon \searrow 0}{\lesssim} \log S(K, \varepsilon) \underset{\varepsilon \searrow 0}{\lesssim} D(K) \log(1/\varepsilon). \quad (3.4)$$

The results presented in [1] are as follows:

$$\begin{aligned}
D(K) &\doteq \overline{\lim}_{\varepsilon \searrow 0} \frac{\log S(K, \varepsilon)}{\log(1/\varepsilon)} = \overline{\lim}_{n \rightarrow \infty} \frac{\log n}{\log(1/\lambda_n)} \\
&= \inf\{p > 0 : \sum_{n=1}^{\infty} \lambda_n^p = \text{Tr}(K^p) < \infty\} \\
&= \inf\{p > 0 : K \in L^p(\mathcal{B}(\mathcal{H}))\}; \tag{3.5}
\end{aligned}$$

$$d(K) \doteq \lim_{\varepsilon \searrow 0} \frac{\log S(K, \varepsilon)}{\log(1/\varepsilon)} = \lim_{\varepsilon \searrow 0} \frac{\log m(K, \varepsilon)}{\log(1/\varepsilon)} = \lim_{n \rightarrow \infty} \frac{\log n}{\log(1/\lambda_n)}, \tag{3.6}$$

where $m(K, \varepsilon) \doteq \max\{n : \lambda_n > \varepsilon\}$ with λ_n being the n 'th largest eigenvalue of the compact operator K :

$$K = \sum_i \lambda_i |\xi_i\rangle \langle \xi_i|. \tag{3.7}$$

b) While the validity of the above Schatten decomposition (3.7) looks to be restricted to the operators in a Hilbert space only, its essence can be carried over through the standard algebraic method of “changes of rings” to far more general contexts of *rigged modules* appearing in the theory of operator spaces, which can be summarised briefly as follows. First, the Banach spaces, \mathfrak{E} and \mathfrak{F} , respectively as the domain and target spaces of Θ , can be replaced more appropriately in our context by the *operator spaces*² as “quantised Banach spaces” whose concrete form can be understood as subspaces of operator algebras, $\mathfrak{E} \subseteq \mathcal{B}(\mathcal{H}_1)$ and $\mathfrak{F} \subseteq \mathcal{B}(\mathcal{H}_2)$, and whose intrinsic characterisation is given in terms of the topology describing the *complete boundedness* in terms of the norm $\|x\|_{cb} \doteq \sup_n \|x\|_n$ which should satisfy the following conditions [12, 21]:

$$\|\alpha \cdot x \cdot \beta\|_n \leq \|\alpha\| \|x\|_n \|\beta\| \quad (\text{for } \forall \alpha, \beta \in M_n(\mathbb{C}), x \in M_n(\mathfrak{E})), \tag{3.8}$$

$$\left\| \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\|_{n+m} \leq \max(\|x\|_n, \|y\|_m) \quad (\text{for } \forall x \in M_n(\mathfrak{E}), \forall y \in M_m(\mathfrak{E})). \tag{3.9}$$

A linear map $\Theta : \mathfrak{E} \rightarrow \mathfrak{F}$ from an operator space \mathfrak{E} to another \mathfrak{F} is completely bounded if it is bounded with respect to the spatial tensor norm,

$$\|\Theta\|_{cb} \doteq \sup_{n \in \mathbb{N}} \|\Theta \otimes id_{M_n}\| < \infty, \tag{3.10}$$

²After this paper was completed, F. Fidaleo kindly drew our attention to his article [15] which seems to be the first appearance of operator space techniques in algebraic QFT. Fidaleo uses non-commutative L^p -spaces to reformulate the split property as a natural consequence of their operator space structures. Our aim here, however, is rather different; we seek an approach to the goals i) and ii) above by means of an extension of ε -entropy to the context of generalised compact operators on the rigged modules.

where $\Theta \otimes \text{id}_{M_n} : \mathfrak{E} \otimes M_n(\mathbb{C}) = M_n(\mathfrak{E}) \rightarrow M_n(\mathfrak{F})$. The total set of completely bounded operators from \mathfrak{E} to \mathfrak{F} is denoted by $CB(\mathfrak{E}, \mathfrak{F})$. To adapt these definitions to our present context, it is more convenient to restrict operator spaces to the *rigged modules* defined as follows:

Definition 3.1 (Blecher [3]). A right A -operator module Y with a (non-selfadjoint) operator algebra A is called a **right A -rigged module** if there is a net of positive integers $n(\beta)$ and right A -module maps $\phi_\beta : Y \rightarrow C_{n(\beta)}(A) = A \otimes M_{n(\beta),1}(\mathbb{C})$ and $\psi_\beta : C_{n(\beta)}(A) \rightarrow Y$ such that

- (1) ϕ_β and ψ_β are completely contractive (i. e., contractive with respect to $\| \cdot \|_{cb}$);
- (2) $\psi_\beta \phi_\beta \rightarrow \text{id}_Y$ strongly on Y ;
- (3) the maps ψ_β are right A -essential (i. e., $\overline{\text{span}(Y \cdot A)} = Y$);
- (4) $\phi_\gamma \psi_\beta \phi_\beta \rightarrow \phi_\gamma$ uniformly in norm for $\forall \gamma$.

What is remarkable about this notion is not only that a Hilbert C^* -module as an operator-module analog of a Hilbert space is a rigged module in this sense, but also that an arbitrary rigged module can be embedded into a (possibly nonunique) Hilbert C^* -module as stated in the following theorem:

Theorem 3.2 (Blecher [3]). *Let A be a (non-selfadjoint) operator algebra with c.a.i. (= contractive approximate identity), B a C^* -algebra generated by the unitalisation A_+ of A and Z a Hilbert C^* -module over B . Suppose that Y is a closed A -submodule of Z and that $W \doteq \{z \in Z : \langle z|y \rangle \in A \text{ for } \forall y \in Y\}$. Suppose further that there exists a c.a.i. for $\mathbb{K}(Z)$ consisting of elements $\sum_{k=1}^n |y_k\rangle\langle w_k|$ in $D \doteq \text{span}\{|y\rangle\langle w| : w \in W, y \in Y\}$ with $\sum_{k=1}^n |y_k\rangle\langle y_k| \leq 1$ and $\sum_{k=1}^n |w_k\rangle\langle w_k| \leq 1$. Let C be the closure of D in $\mathbb{K}(Z)$. Then Y is a right A -rigged module and W is a left A -rigged module. Moreover, $\tilde{Y} \cong \{\bar{w} \in \bar{Z} : w \in W\}$ as A -operator modules (completely isometrically) and $\mathbb{K}(Y) \cong C$ completely isometrically isomorphically. Conversely, every rigged module arises in this way.*

In the above, the algebra $\mathbb{K}(Y)$ of generalised compact operators is defined by the norm limits in $\| \cdot \|_{cb}$ of finite-rank operators given by the linear combinations of $|y\rangle\langle f|$, $y \in Y$, $f \in \tilde{Y} \doteq \{f \in CB(Y, A) : f \text{ is } A\text{-linear and } (\psi_\beta \phi_\beta)^* f \rightarrow f \text{ uniformly}\}$, where \tilde{Y} is the complete-boundedness analog of the dual of Y . In terms of the Haagerup module tensor product \otimes_{hA} [12, 21] (as the most appropriate definition of tensor products in the contexts of operator spaces and of Hilbert modules), the natural relation $\mathbb{K}(Y) \cong Y \otimes_{hA} \tilde{Y}$ holds. In this context our nuclear map Θ should belong to $\mathbb{K}(\mathfrak{E}, \mathfrak{F})$ which is also a rigged module and which can be embedded into the *linking algebra* $\mathbb{K}(\mathfrak{E} \oplus \mathfrak{F})$ as its “corner” entity:

$$\mathbb{K}(\mathfrak{E}, \mathfrak{F}) = \mathfrak{F} \otimes_{hA} \tilde{\mathfrak{E}} \hookrightarrow \mathbb{K}(\mathfrak{E} \oplus \mathfrak{F}) = \begin{pmatrix} \mathbb{K}(\mathfrak{E}) & \mathbb{K}(\mathfrak{F}, \mathfrak{E}) \\ \mathbb{K}(\mathfrak{E}, \mathfrak{F}) & \mathbb{K}(\mathfrak{F}) \end{pmatrix}. \quad (3.11)$$

While the generalised compact operators are not compact operators in the genuine sense, they share many important features of the latter allowing the finite-dimensional approximations, which constitutes the essential ingredients of the ‘riggedness’. In this way, the essence of the Schatten decomposition $K = \sum_i \lambda_i |\xi_i\rangle\langle\xi_i|$ can be recovered and generalised in the form of operator partition of unity: $\Theta = \sum_i \lambda_i |x_i\rangle\langle y_i|$ on the basis of which a variety of *entropy-like quantities* can be defined and calculated as already indicated by the above discussion of ε -entropy (cf. Alicki’s formulation of non-commutative dynamical entropy). Then, the essence of the quantum energy inequalities might perhaps be formulated as the stability condition imposed on the vacuum-like states in relation to the Legendre transform involving the energy (density) and one of the suitable entropy-like quantities (e.g., α -divergence and relative entropy [2]) which essentially originate from the type-III property of local subalgebras appearing in algebraic QFT.

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